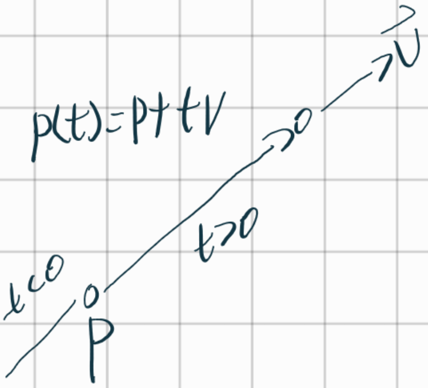
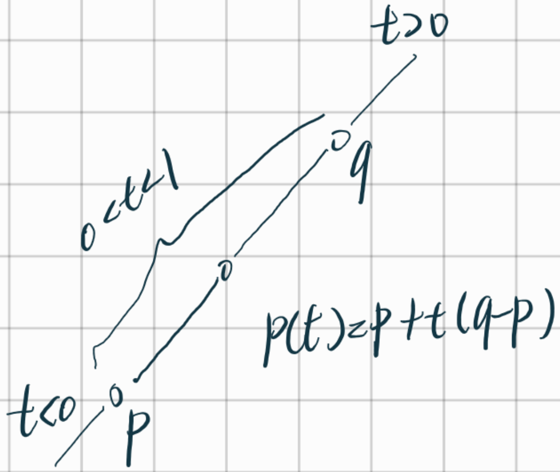
Example 1: Linear Representation 线性表示





t 表示时间

t 是一个插值

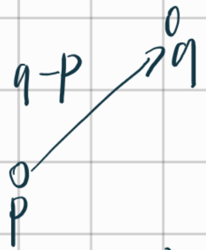
Vector Norm 矢量大小

$\|p\|_2 = (p_x^2+p_y^2+p_z^2)^{\frac{1}{2}}$

欧几里得 norm (模)

$\|\mathbf{p}\|_p = \left(\|p_x\|^p + \|p_y\|^p + \|p_z\|^p\right)^{1/p}$	p-norm
$\|\mathbf{p}\|_1 = \|p_x\| + \|p_y\| + \|p_z\|$	1-norm
$\|\mathbf{p}\|_\infty = \max(\|p_x\|, \|p_x\|, \|p_x\|)$	Infinity norm

Verctor Norm: Usage 用法



$\|q-p\|$

q到p的距离

$\|p\|=1$

单位向量

$\hat{p} = p/\|p\|$

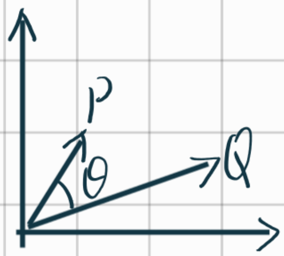
单位化(归一化)

Vector Arithematic: Dot Product 点乘(内积)

在数学上:

$$p\cdot q = p_x q_x + p_y q_y + p_z q_z = p^T q$$

$$= \|p\|\|q\|\cos\theta$$

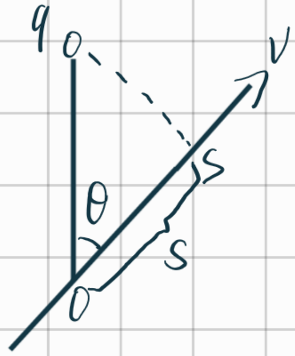


运算归则:

- $\mathbf{p}\cdot\mathbf{q} = \mathbf{q}\cdot\mathbf{p}$
- $\mathbf{p}\cdot(\mathbf{q}+\mathbf{r}) = \mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}$
- $\mathbf{p}\cdot\mathbf{p} = \|\mathbf{p}\|_2^2$, a different way to write norm.

◦ 当 $p\cdot q = 0$, p、q 正交

Example 2: Particle-Line Projection (部分线投影)



$$s = \|q-o\|\cos\theta$$

$$s = \boxed{\|q-o\|\,\|v\|\cos\theta} / \|v\|$$

$$\Updownarrow$$

$$(q-o)\cdot v \iff (q-o)^T v$$

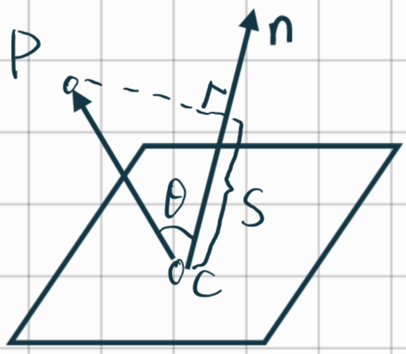
$$s = (q-o)^T v / \|v\|$$

$$= (q-o)^T \bar{v}$$

And

$$\vec{s} = o + s\bar{v}$$

Example 3: 平面表示



$$S = \boxed{(p-c)^T n} \begin{cases} >0 & \text{平面以上 above} \\ =0 & \text{平面中 in} \\ <0 & \text{平面下 below} \end{cases}$$

$$\Updownarrow$$

$$|p-o|\cos\theta$$

作用：

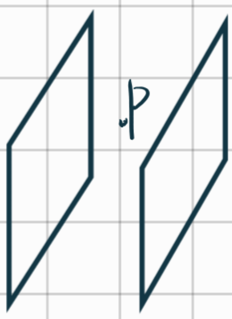
① 做碰撞检测

思考题：

判断一个点是否在盒子里

解答：

这里可以回想Games101中的AABB，



一个盒子由六个面构成，只要这个点满足：

对一面 $(p-o)^T \cdot n > 0$

对另一面 $(p-o)^T \cdot n < 0$

且这三个相对的面都满足，那么点 P 在盒子内

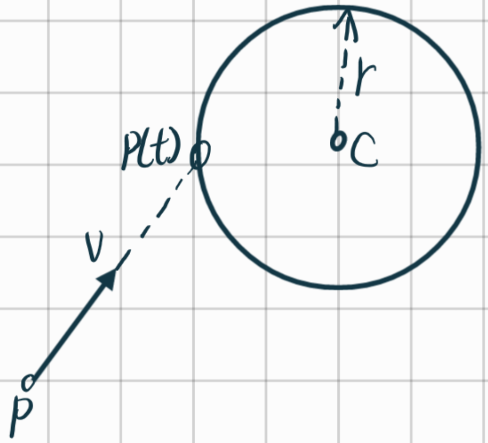
Example4：粒子-球的碰撞

由例1： $p(t) = p + tv$

$$\|p(t) - c\|^2 = r^2$$

$$(p - c + tv)(p - c + tv) = r^2$$

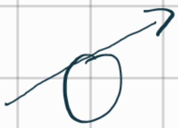
$$v \cdot v t^2 + 2(p-c) \cdot v t + (p-c) \cdot (p-c) - r^2 = 0$$



三个结果：

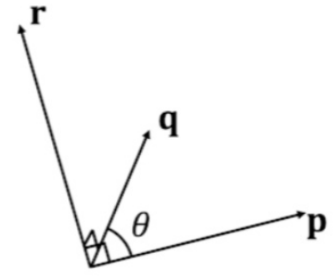
无根

一个根



两个根

Vector Arithematic：叉乘(外积)



Geometric Meanings

$$r = p \times q = \begin{bmatrix} p_y q_z - p_z q_y \\ p_z q_x - p_x q_z \\ p_x q_y - p_y q_x \end{bmatrix}$$

$$= \det \begin{vmatrix} i & j & k \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}$$

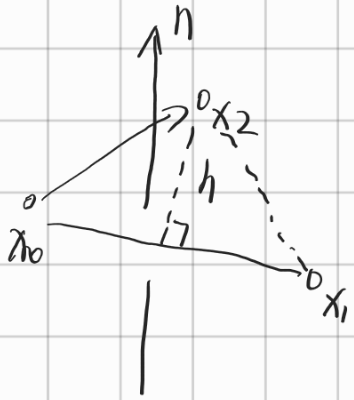
$r \cdot p = 0$; $r \cdot q = 0$; $\|r\| = \|p\| \|q\| \sin\theta$

$p \times q = -q \times p$

$p \times (q + r) = p \times q + p \times r$

当 $p \times q = 0$ 那么 p、q 平行

Example5: Triangle Normal and Area 三角形的法线和面积



边：$x_{10}=x_1-x_0 \quad x_{20}=x_2-x_0$

法线：$n=(x_{10}\times x_{20})/\|x_{10}\times x_{20}\|$

面积：$A=\underbrace{\|x_{10}\| \quad h/2}_{底 \times 高/2} = \|x_{10}\|\|x_{20}\|\sin\theta/2$

$\|x_{10}\times x_{20}\|$

$=\|x_{10}\times x_{20}\|/2$

通过这个例子：

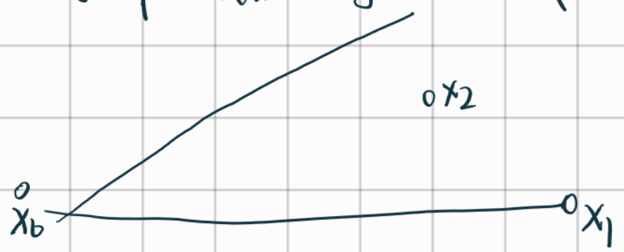
由两个向量的叉乘可知：

1、一个面的法向量

2、一个面的面积

思考题：

如何判断三个点在一条直线上？



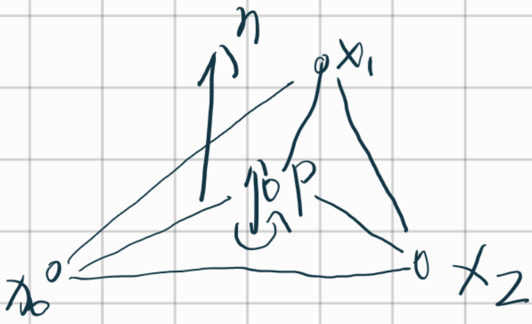
方法1：求得面积为0则在一条直线上

计 $A=\|x_{10}\times x_{20}\|/2=0$ then they are on the same line

也可 $\underset{\sim\sim\sim\sim}{x_{10}\times x_{20}=0}$

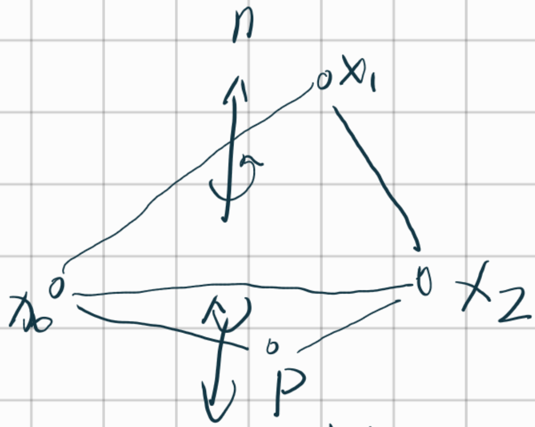
Example 6 Triangle Inside/outside Test

（测试某一点在三角形中或外）



当P在 x_0x_1 内侧

$$(x_0-p)\times(x_1-p)\cdot n>0$$

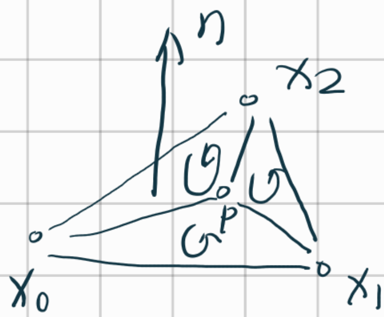


当P在 $x_0\,x_1$ 外侧

$$(x_0-p)\times(x_1-p)\cdot n<0$$

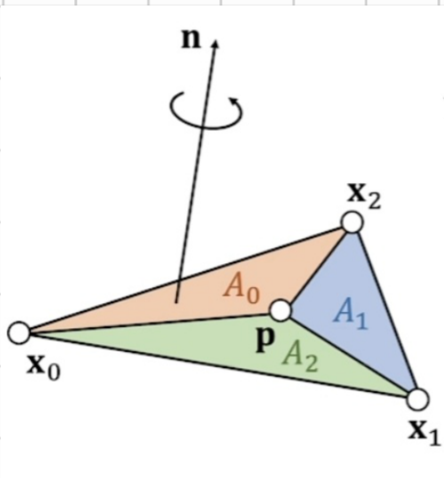
总结：

如果 P在三条边的内侧，即：



$$\left.\begin{aligned}(x_0-p)\times(x_1-p)\cdot n>0\\(x_1-p)\times(x_2-p)\cdot n>0\\(x_2-p)\times(x_0-p)\cdot n>0\end{aligned}\right\}\text{在三角形内}$$

Example 7: Barycentric Coordinates (重心坐标)



当P点在三角形中，

$$\frac{1}{2}(x_0-p)\times(x_1-p)\cdot n = \begin{cases} \frac{1}{2}\|(x_0-p)\times(x_1-p)\| & 内 \\ -\frac{1}{2}\|(x_0-p)\times(x_1-p)\| & 外 \end{cases}$$

面积：

$$A_0 = \frac{1}{2}(x_0-p)\times(x_1-p)\cdot n$$

$$A_1 = \frac{1}{2}(x_1-p)\times(x_2-p)\cdot n$$

$$A_2 = \frac{1}{2}(x_2-p)\times(x_0-p)\cdot n$$

$$A = A_0 + A_1 + A_2$$

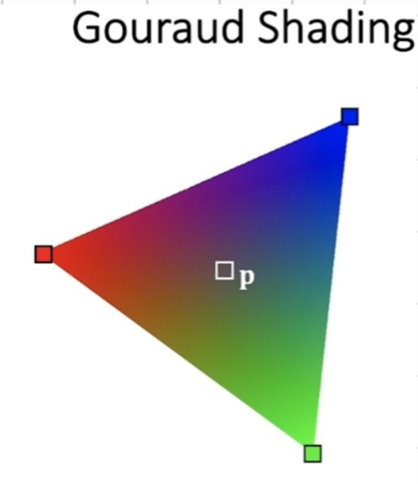
重心权重： $b_0 = A_0/A$ $b_1 = A_1/A$ $b_2 = A_2/A$

α β γ

$$b_0 + b_1 + b_2 = 1$$

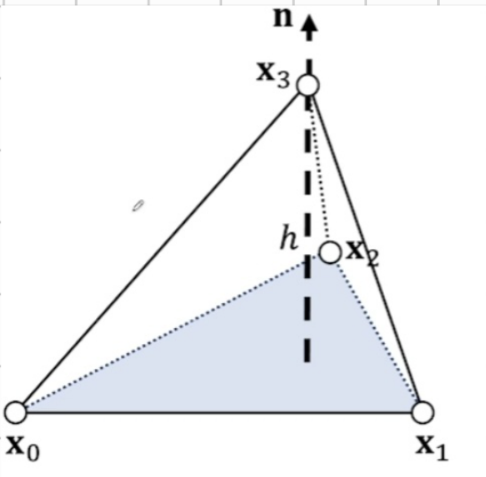
重心插值： $p = b_0 x_0 + b_1 x_1 + b_2 x_2$

Gouraud shading (平滑着色) Gouraud只是发明者名字



为三个点的颜色作插值

Excample 9 Tetrahedral Volume



边：

$x_{10}=x_1-x_0 \quad x_{20}=x_2-x_0 \quad x_{30}=x_3-x_0$

底面积

$$A=\frac{1}{2}\|x_{10}\times x_{20}\|$$

高

$$h=x_{30}\cdot n=x_{30}\cdot\frac{x_{10}\times x_{20}}{\|x_{10}\times x_{20}\|}$$

x_{30}在n方向的投影

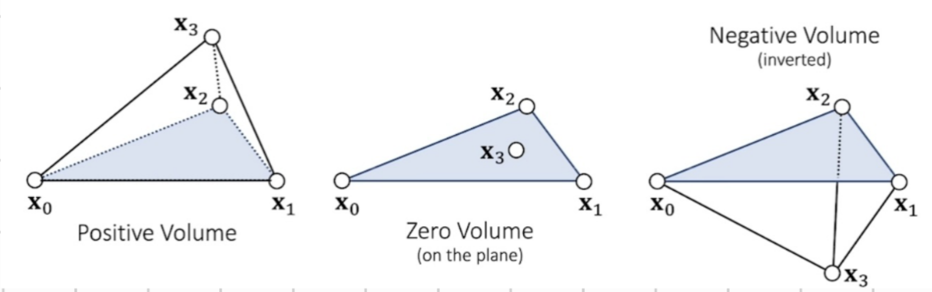
体积：

$$V=\frac{1}{3}hA=\frac{1}{3}\boxed{x_{30}\cdot\frac{x_{10}\times x_{20}}{\|x_{10}\times x_{20}\|}}\cdot\boxed{\frac{1}{2}\|x_{10}\times x_{20}\|}$$

$$=\frac{1}{6}x_{30}\cdot x_{10}\times x_{20}$$

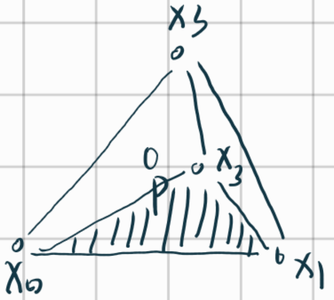
$$=\frac{1}{6}\begin{vmatrix} x_1 & x_2 & x_3 & x_0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Note that the volume $V=\frac{1}{3}hA=\frac{1}{6}\,\mathbf{x}_{30}\cdot\mathbf{x}_{10}\times\mathbf{x}_{20}$ is signed.



这样算出来的体积都是带符号的

Excample 10： Barycentric Weight (重心权重)



P与四个顶点将四面体分成了四部分

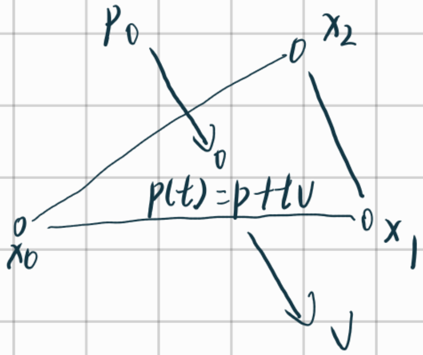
$$V_0 = Vol(x_3, x_2, x_1, p)$$
$$V_1 = Vol(x_3, x_2, x_0, p)$$
$$V_2 = Vol(x_3, x_1, x_0, p)$$
$$V_3 = Vol(x_2, x_1, x_0, p)$$

当P在四面体中时 $V_0, V_1, V_2, V_3 > 0$

重心权重

$b_0 = V_0/V \quad b_1 = V_1/V \quad b_2 = V_2/V \quad b_3 = V_3/V$

Example 11： 粒子-三角形 碰撞



当点与三角形相遇时

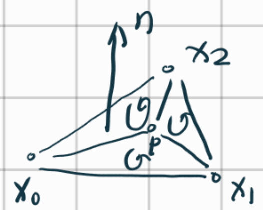
① 根据上一个例子 体积公式 可知

$$(p(t) - x_0)\cdot x_{10} \times x_{20} = 0$$
$$(p - x_0 + tv)\cdot x_{10} \times x_{20} = 0$$
$$t = \frac{(p - x_0)\cdot x_{10} \times x_{20}}{v \cdot x_{10} \times x_{20}}$$

② 再检测 p(t)是否在三角形内侧 由Example 6

如果 P在三条边的内侧，即：

$$\left.\begin{aligned} (x_0 - p)\times(x_1 - p)\cdot n > 0 \\ (x_1 - p)\times(x_2 - p)\cdot n > 0 \\ (x_2 - p)\times(x_0 - p)\cdot n > 0 \end{aligned}\right\} \text{在三角形内}$$



Matrix: Defination 矩阵定义

A *real* matrix is a set of *real* elements arranged in rows and columns.

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = [\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2] \in \mathbf{R}^{3\times3}$$

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{00} & a_{10} & a_{20} \\ a_{01} & a_{11} & a_{21} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}$$

Transpose

$$\begin{bmatrix} a_{00} & & \\ & a_{11} & \\ & & a_{22} \end{bmatrix}$$

Diagonal

$$\mathbf{I} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Identity

$\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ Symmetric

转置　　对角矩阵　　单位矩阵

Matrix: Mulitplication

$$AB \neq BA \qquad (AB)x = A(Bx)$$

$$(AB)^T = B^T A^T \qquad (A^T A)^T = A^T A$$

$$Ix = x \qquad AI = IA = A$$

$$A^{-1}: \; AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Matrix: Orthogonality 正交矩阵

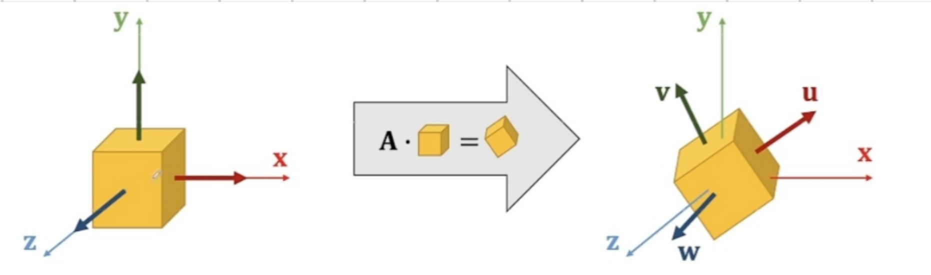
正交矩阵乘以它的转置

$A = [a_0 \; a_1 \; a_2]$ 当 $a_i^T a_j = \begin{cases} 1 & 当 i=j \\ 0 & 当 i\neq j \end{cases}$

$$A^T A = \begin{bmatrix} a_0^T \\ a_1^T \\ a_2^T \end{bmatrix} [a_0 \; a_1 \; a_2] = \begin{bmatrix} a_0^T a_0 & a_0^T a_1 & a_0^T a_2 \\ a_1^T a_0 & a_1^T a_1 & a_1^T a_2 \\ a_2^T a_0 & a_2^T a_1 & a_2^T a_2 \end{bmatrix} = I$$

$$A^T = A^{-1}$$

Matrix Transformation 矩阵变换



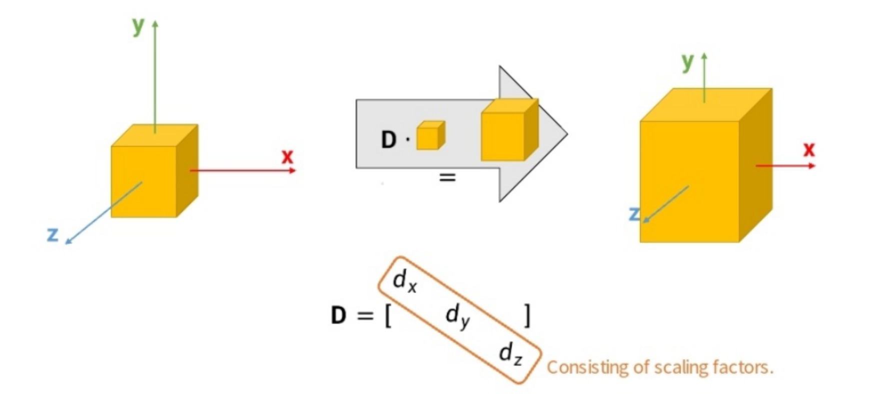
MVP 变换中的变换

$$I = \begin{bmatrix} x & y & z \end{bmatrix}$$

$$\begin{aligned} Ax &= u \\ Ay &= v \quad \Rightarrow A = [u\ v\ w] \\ Az &= w \end{aligned}$$

A是正交矩阵

A scaling can be represented by a diagonal matrix.



MVP中的M

Singular Value Decomposition 奇异值分解

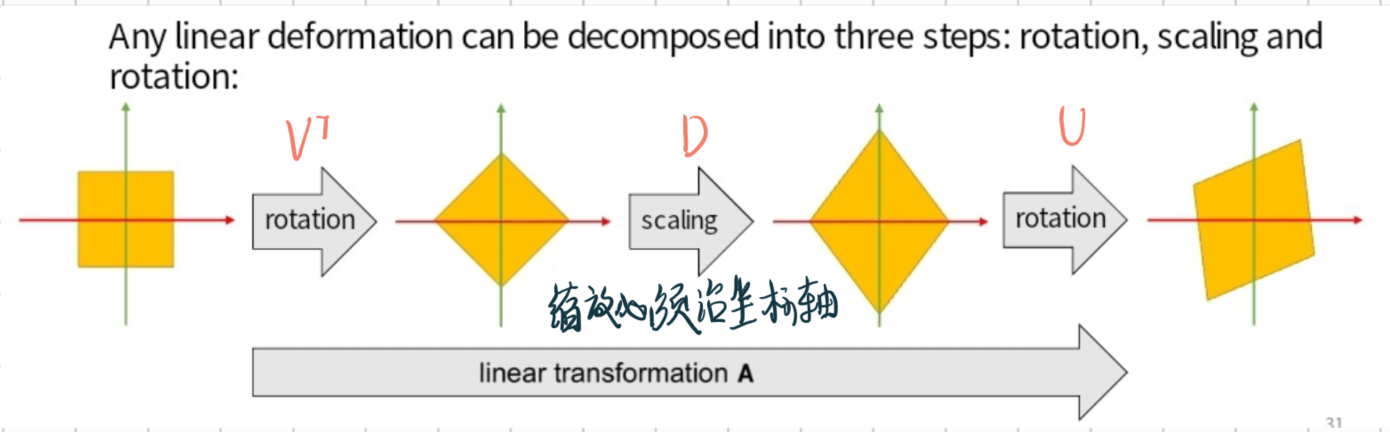
一个矩阵可分为三个部分

$A=UDV^T$ D：奇异值

U and V 是正交值

在图形学中这样解释：

一个线性变换一定可以被拆分成三个部分：



Eigenvalue Decomposition 特征值分解

$A=UDU^T$ D：特征值

UDU^{-1} U：正交值

Symmetric Positive Definiteness (s.p.d) 对称正定

$d>0 \iff v^T d v > 0$ 对任意 $v \neq 0$

$d_0, d_1, \ldots > 0 \iff v^T D v = v^T \begin{bmatrix} \ddots & & \\ & d_i & \\ & & \ddots \end{bmatrix} v > 0$

$d_0, d_1, \ldots > 0 \iff v^T (UDU^T) v = v^T U U^T (UDU^T) U U^T v$

对角矩阵对角上都是正数的矩阵

判断正定：

$a_{ii} > \Sigma_{i \neq j} |a_{ij}|$ for all i

对于对角上的元素大于该行所有该行元素绝对值的和，

称之为正定矩阵 P.d

$\begin{bmatrix} 4 & 3 & 0 \\ -1 & 5 & 3 \\ -8 & 0 & 9 \end{bmatrix}$ $\quad 4 > 3 + 0$, $5 > 1 + 3$, $8 > 0 + 9$

对称正定矩阵一定是可逆的：

$A^{-1} = (U^T)^{-1} D^{-1} U^{-1} = U D^{-1} U^T$

问题：证明 A是 s.p.d $B=\begin{bmatrix} A & -A \\ -A & A \end{bmatrix}$ 是半正定

对任意 x,y，得知

$$[x^T\ y^T]\,B\begin{bmatrix} x \\ y \end{bmatrix} = [x^T\ y^T]\begin{bmatrix} A & -A \\ -A & A \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [Ax^T - Ay^T \quad -Ax^T + Ay^T]\begin{bmatrix} x \\ y \end{bmatrix}$$

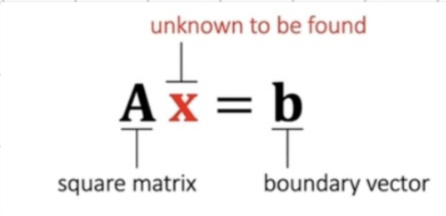
$$= Ax^T(x-y) - Ay^T(x-y)$$

$$= (x-y)^T A (x-y)$$

因为 A 正定，所以 $(x-y)^T A (x-y) \geq 0$

所以 B 是半正定

LinearSolver 线性系统



本质就是求 A^{-1} $\Leftrightarrow$ $x = A^{-1}b$

但 A^{-1} 是难以求得的

Direct Linear Solver 直接解法

基于LU分解

$$A = LU = \begin{bmatrix} L_{00} & & \\ L_{10} & L_{11} & \\ \vdots & \cdots & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \cdots & \vdots \\ & u_{n-1,n-1} & \vdots \\ & & u_{nn} \end{bmatrix}$$

下三角矩阵　　上三角矩阵

① 解 $Ly = b$

$$\begin{bmatrix} l_{00} & & \\ l_{10} & l_{11} & \\ \vdots & \cdots & \ddots \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \end{bmatrix}$$

$l_{10} y_1 + l_{11} y_1 = b_1$

$(l_{10} + l_{20}) y_1 = b_1$

$l_{00} y_0 = b_0$

$y_0 = b_0 / l_{00}$

$y_1 = (b_1 - l_{10} y_0) / l_{11}$

$\cdots$

② 解: $Ux=y$

$$\begin{bmatrix} \ddots & \cdots & \vdots \\ & u_{n-1,n-1} & \vdots \\ & & u_{n,n} \end{bmatrix} \begin{bmatrix} \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

$x_n = y_n / u_n$

$x_{n-1} = (y_{n-1} - u_{n-1,n} x_n) / u_{n-1,n-1}$

$\cdots$

总结:

当A是稀疏的,但L和U可能不是稀疏的

Iterative Linear Solver 迭代法

$$x^{[k+1]} = x^{[k]} + \alpha \underbrace{M^{-1}}_{\text{迭代矩阵}} \underbrace{(b - Ax^{[k]})}_{\text{残差}}$$

$$b - Ax^{[k+1]} = b - Ax^{[k]} - \alpha M^{-1}(b - Ax^{[k]})$$
$$= (I - \alpha AM^{-1})(b - Ax^{[k]}) = (I - \alpha AM^{-1})^{k+1}(b - Ax^{[0]})$$

所以

$b - Ax^{[k+1]} \to 0$, if $\rho(I - \alpha AM) < 1$

看不懂要死了

Iterative Linear Solver

An iterative solver has the form:

$$\mathbf{x}^{[k+1]} = \mathbf{x}^{[k]} + \overbrace{\alpha}^{\text{relaxation}} \underbrace{\mathbf{M}^{-1}}_{\text{iterative matrix}} \underbrace{(\mathbf{b} - \mathbf{A}\mathbf{x}^{[k]})}_{\text{residual error}}$$

M must be easier to solve:

$\mathbf{M} = \mathrm{diag}(\mathbf{A})$ Jacobi Method	$\mathbf{M} = \mathrm{lower}(\mathbf{A})$ Gauss-Seidel Method

The convergence can be accelerated: Chebyshev, Conjugate Gradient, ...
(Omitted here.)

simple | fast for inexact solution | parallelable | convergence condition | slow for exact solution

Tensor Calculus 微积分

Basic Concepts: 1st-Order Derivatives 阶导

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \underline{\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{bmatrix}}\begin{bmatrix}dx\\dy\\dz\end{bmatrix}$$

梯度

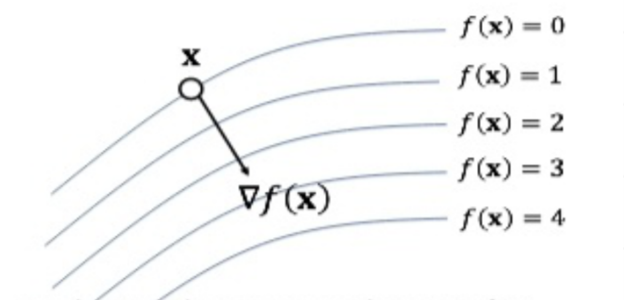
$$\Downarrow$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{bmatrix}$$

or

$$\nabla f(x) = \begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{bmatrix}$$

gradient 梯度



Gradient is the steepest direction for increasing f. It's perpendicular to the isosurface.

梯度是垂直于等高线的方向

函数值最快上升的方向

if $f(x) = \begin{bmatrix}f(x)\\g(x)\\h(x)\end{bmatrix} \in R^3$, then

Jacobian 矩阵

$$J(x) = \frac{\partial f}{\partial x} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}\\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z}\end{bmatrix}$$

散度，对角上的和

$$\nabla \cdot f = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Curl

$$\nabla \times f = \begin{bmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{bmatrix}$$

与流体有关

Basic Concepts: 2nd-Order Derivatives 二阶导

$$H = J(\nabla f(x)) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

$$\nabla \cdot \nabla f(x) = \nabla^2 \cdot f(x)$$
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

拉普拉斯方程

Taylor Expansion 泰勒展开

if $f(x) \in R$ then:

$$f(x) = f(x_0) + \frac{\partial f(x_0)}{\partial x}(x - x_0) + \frac{1}{2}\frac{\partial f^2(x_0)}{\partial x^2}(x - x_0)^2 + \cdots$$

if $(\vec{x}) \in R$, then:

$$\begin{bmatrix} x\,(x-x_0) \\ y\,(x-x_0) \\ z\,(x-x_0) \end{bmatrix}$$

$$\Updownarrow$$

$$f(x) = f(x_0) + \frac{\partial f(x_0)}{\partial x}(x - x_0) + \frac{1}{2}(x - x_0)^T \frac{\partial f^2(x_0)}{\partial x^2}(x - x_0) + \cdots$$

$$\Updownarrow$$

$$\begin{bmatrix} \frac{\partial f(x_0)}{\partial x} & \frac{\partial f(x_0)}{\partial y} & \frac{\partial f(x_0)}{\partial z} \end{bmatrix}$$

$$[\quad]\,[\quad]\,[\quad]$$

正定矩阵

$$= f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T H (x - x_0) + \cdots$$

问题

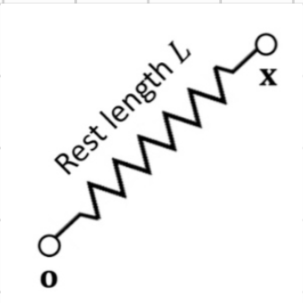
$$\frac{\partial \|\vec{x}\|}{\partial x} = ?$$

$$\frac{\partial \|\vec{x}\|}{\partial \vec{x}} = \frac{\partial (\vec{x}^T\vec{x})^{\frac{1}{2}}}{\partial x} = \frac{1}{2}\underbrace{(\vec{x}^T\vec{x})^{-\frac{1}{2}}}_{\Updownarrow\ \frac{1}{\|\vec{x}\|}}\underbrace{\frac{\partial(\vec{x}^T\vec{x})}{\partial \vec{x}}}_{\Updownarrow} = \frac{1}{2\|\vec{x}\|}2x^T = \frac{x^T}{\|x\|}$$

$$\frac{\partial(\vec{x}^T\vec{x})}{\partial \vec{x}} = \frac{\partial(x^2+y^2+z^2)}{\partial \vec{x}} = [2x \quad 2y \quad 2z] = 2\vec{x}^T$$

$$\therefore \frac{\partial \|\vec{x}\|}{\partial \vec{x}} = \frac{\vec{x}^T}{\|\vec{x}\|}$$

Example: A Spring



Energy:

$$E(x) = \frac{k}{2}(\|\vec{x}\| - L)^2$$

Force

$$f(x) = -\nabla E(x) = \frac{\partial E(\vec{x})}{\partial \vec{x}} = -k(\|\vec{x}\| - L)\left(\frac{\partial \|\vec{x}\|}{\partial \vec{x}}\right)^T \Rightarrow \text{梯度}[\ ]$$

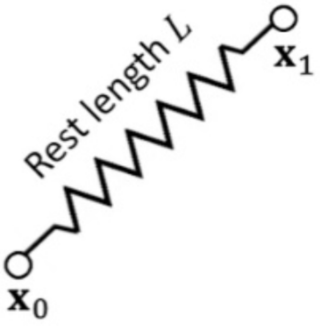
$$= -k(\|\vec{x}\| - L)\frac{\vec{x}}{\|\vec{x}\|}$$

Tangent stiffness:

$$H(x) = -\frac{\partial f(\vec{x})}{\partial \vec{x}} = k\frac{\vec{x}\vec{x}^T}{\|\vec{x}\|^2} + k(\|\vec{x}\| - L)\frac{1}{\|\vec{x}\|} - k(\|\vec{x}\| - L)\frac{\vec{x}}{\|\vec{x}\|}\frac{\vec{x}^T}{\|\vec{x}\|}$$

$$= k\frac{\vec{x}\vec{x}^T}{\|\vec{x}\|} + k\left(1 - \frac{L}{\|\vec{x}\|}\right)\left[I - \frac{\vec{x}\vec{x}^T}{\|\vec{x}\|}\right]$$

Example: A Spring with Two Ends　　没看明白 将往后学吧



$$\mathbf{x}_{01} = \mathbf{x}_0 - \mathbf{x}_1$$

$$\frac{\partial \|\mathbf{x}\|}{\partial \mathbf{x}} = \frac{\mathbf{x}^{\mathrm{T}}}{\|\mathbf{x}\|}$$

Energy:

$$E(\mathbf{x}) = \frac{k}{2}(\|\mathbf{x}_{01}\| - L)^2$$

Force:

$$\mathbf{f}(\mathbf{x}) = -\nabla E(\mathbf{x}) = \begin{bmatrix} -\nabla_0 E(\mathbf{x}) \\ -\nabla_1 E(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e \\ -\mathbf{f}_e \end{bmatrix}$$

$$\mathbf{f}_e = -k(\|\mathbf{x}_{01}\| - L)\frac{\mathbf{x}_{01}}{\|\mathbf{x}_{01}\|}$$

Tangent stiffness:

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 E}{\partial \mathbf{x}_0^2} & \frac{\partial^2 E}{\partial \mathbf{x}_0 \partial \mathbf{x}_1} \\ \frac{\partial^2 E}{\partial \mathbf{x}_0 \partial \mathbf{x}_1} & \frac{\partial^2 E}{\partial \mathbf{x}_1^2} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_e & -\mathbf{H}_e \\ -\mathbf{H}_e & \mathbf{H}_e \end{bmatrix}$$

$$\mathbf{H}_e = k\frac{\mathbf{x}_{01}\mathbf{x}_{01}^{\mathrm{T}}}{\|\mathbf{x}_{01}\|^2} + k\left(1 - \frac{L}{\|\mathbf{x}_{01}\|}\right)\left(\mathbf{I} - \frac{\mathbf{x}_{01}\mathbf{x}_{01}^{\mathrm{T}}}{\|\mathbf{x}_{01}\|^2}\right)$$